1.5 Lines and Planes

Now that we know how to perform some operations on vectors, we can start to deal with some familiar geometric objects, like lines and planes, in the language of vectors. The reason for doing this is simple: using vectors makes it easier to study objects in 3-dimensional Euclidean space. We will first consider lines.

Line through a point, parallel to a vector

Let $P = (x_0, y_0, z_0)$ be a point in $\mathbb{R}^3$, let $\mathbf{v} = (a, b, c)$ be a nonzero vector, and let L be the line through P which is parallel to $\mathbf{v}$ (see Figure 1.5.1).

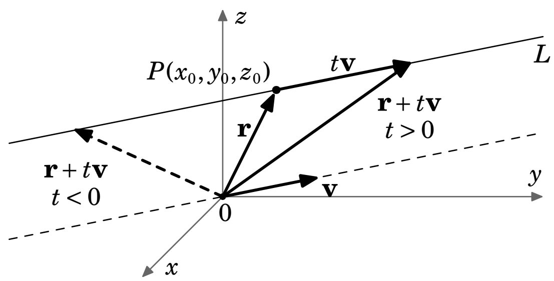


Figure 1.5.1

Let $\mathbf{r} = (x_0, y_0, z_0)$ be the *vector* pointing from the origin to P. Since multiplying the vector $\mathbf{v}$ by a scalar t lengthens or shrinks $\mathbf{v}$ while preserving its direction if $t > 0$, and reversing its direction if $t < 0$, then we see from Figure 1.5.1 that every point on the line L can be obtained by adding the vector $t\mathbf{v}$ to the vector $\mathbf{r}$ for some scalar t. That is, as t varies over all real numbers, the vector $\mathbf{r} + t\mathbf{v}$ will point to every point on L. We can summarize the *vector representation of L* as follows:

For a point $P = (x_0, y_0, z_0)$ and nonzero vector $\mathbf{v}$ in $\mathbb{R}^3$, the line L through P parallel to $\mathbf{v}$ is given by

$$\mathbf{r} + t\mathbf{v}, \quad \text{for } -\infty < t < \infty \tag{1.16}$$

where $\mathbf{r} = (x_0, y_0, z_0)$ is the vector pointing to P.

Note that we used the correspondence between a vector and its terminal point. Since $\mathbf{v} = (a, b, c)$, then the terminal point of the vector $\mathbf{r} + t\mathbf{v}$ is $(x_0 + at, y_0 + bt, z_0 + ct)$. We then get the *parametric representation of L* with the *parameter t*:

For a point $P = (x_0, y_0, z_0)$ and nonzero vector $\mathbf{v} = (a, b, c)$ in $\mathbb{R}^3$, the line L through P parallel to $\mathbf{v}$ consists of all points (x, y, z) given by

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct, \quad \text{for } -\infty < t < \infty \tag{1.17}$$

Note that in both representations we get the point P on L by letting $t = 0$.

In formula (1.17), if $a \neq 0$, then we can solve for the parameter t: $t = (x - x_0)/a$. We can also solve for t in terms of y and in terms of z if neither b nor c, respectively, is zero: $t = (y - y_0)/b$ and $t = (z - z_0)/c$. These three values all equal the same value t, so we can write the following system of equalities, called the *symmetric representation of* L:

For a point $P = (x_0, y_0, z_0)$ and vector $\mathbf{v} = (a, b, c)$ in $\mathbb{R}^3$ with a, b and c all nonzero, the line L through P parallel to $\mathbf{v}$ consists of all points (x, y, z) given by the equations

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \tag{1.18}$$

What if, say, $a = 0$ in the above scenario? We can not divide by zero, but we do know that $x = x_0 + at$, and so $x = x_0 + 0t = x_0$. Then the symmetric representation of L would be:

$$x = x_0, \quad \frac{y - y_0}{b} = \frac{z - z_0}{c} \tag{1.19}$$

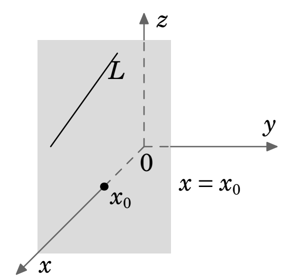


Figure 1.5.2

Note that this says that the line L lies in the *plane* $x = x_0$, which is parallel to the yz-plane (see Figure 1.5.2). Similar equations can be derived for the cases when $b = 0$ or $c = 0$.

You may have noticed that the vector representation of L in formula (1.16) is more compact than the parametric and symmetric formulas. That is an advantage of using vector notation. Technically, though, the vector representation gives us the *vectors* whose terminal points make up the line L, not just L itself. So you have to remember to identify the vectors $\mathbf{r} + t\mathbf{v}$ with their terminal points. On the other hand, the parametric representation *always* gives just the points on L and nothing else.

Example 1.19. Write the line L through the point $P = (2, 3, 5)$ and parallel to the vector $\mathbf{v} = (4, -1, 6)$, in the following forms: (a) vector, (b) parametric, (c) symmetric. Lastly: (d) find two points on L distinct from P.

Solution: (a) Let $\mathbf{r} = (2, 3, 5)$. Then by formula (1.16), L is given by:

$$\mathbf{r} + t\mathbf{v} = (2, 3, 5) + t(4, -1, 6), \quad \text{for } -\infty < t < \infty$$

(b) L consists of the points (x, y, z) such that

$$x = 2 + 4t, \quad y = 3 - t, \quad z = 5 + 6t, \quad \text{for } -\infty < t < \infty$$

(c) L consists of the points (x, y, z) such that

$$\frac{x - 2}{4} = \frac{y - 3}{-1} = \frac{z - 5}{6}$$

(d) Letting $t = 1$ and $t = 2$ in part(b) yields the points $(6, 2, 11)$ and $(10, 1, 17)$ on L.

Line through two points

Let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ be distinct points in $\mathbb{R}^3$, and let L be the line through P_1 and P_2. Let $\mathbf{r}_1 = (x_1, y_1, z_1)$ and $\mathbf{r}_2 = (x_2, y_2, z_2)$ be the vectors pointing to P_1 and P_2, respectively. Then as we can see from Figure 1.5.3, $\mathbf{r}_2 - \mathbf{r}_1$ is the vector from P_1 to P_2. So if we multiply the vector $\mathbf{r}_2 - \mathbf{r}_1$ by a scalar t and add it to the vector $\mathbf{r}_1$, we will get the entire line L as t varies over all real numbers. The following is a summary of the vector, parametric, and symmetric forms for the line L:

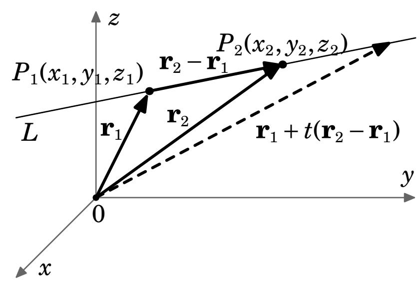


Figure 1.5.3

Let $P_1 = (x_1, y_1, z_1)$, $P_2 = (x_2, y_2, z_2)$ be distinct points in $\mathbb{R}^3$, and let $\mathbf{r}_1 = (x_1, y_1, z_1)$, $\mathbf{r}_2 = (x_2, y_2, z_2)$. Then the line L through P_1 and P_2 has the following representations:

Vector:

$$\mathbf{r}_1 + t(\mathbf{r}_2 - \mathbf{r}_1)\ , \ \text{ for } -\infty < t < \infty \tag{1.20}$$

Parametric:

$$x = x_1 + (x_2 - x_1)t, \quad y = y_1 + (y_2 - y_1)t, \quad z = z_1 + (z_2 - z_1)t, \ \text{ for } -\infty < t < \infty \tag{1.21}$$

Symmetric:

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} \quad (\text{if } x_1 \neq x_2,\ y_1 \neq y_2, \text{ and } z_1 \neq z_2) \tag{1.22}$$

Example 1.20. Write the line L through the points $P_1 = (-3, 1, -4)$ and $P_2 = (4, 4, -6)$ in parametric form.

Solution: By formula (1.21), L consists of the points (x, y, z) such that

$$x = -3 + 7t, \quad y = 1 + 3t, \quad z = -4 - 2t, \ \text{ for } -\infty < t < \infty$$

Distance between a point and a line

Let L be a line in $\mathbb{R}^3$ in vector form as $\mathbf{r} + t\mathbf{v}$ (for $-\infty < t < \infty$), and let P be a point not on L. The distance d from P to L is the length of the line segment from P to L which is perpendicular to L (see Figure 1.5.4). Pick a point Q on L, and let $\mathbf{w}$ be the vector from Q to P. If θ is the angle between $\mathbf{w}$ and $\mathbf{v}$, then $d = \|\mathbf{w}\| \sin\theta$. So since $\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\|\,\|\mathbf{w}\| \sin\theta$ and $\mathbf{v} \neq \mathbf{0}$, then:

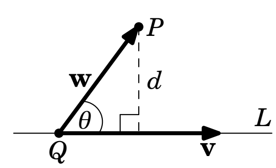


Figure 1.5.4

$$d = \frac{\|\mathbf{v} \times \mathbf{w}\|}{\|\mathbf{v}\|} \tag{1.23}$$

Example 1.21. Find the distance d from the point $P = (1,1,1)$ to the line L in Example 1.20.

Solution: From Example 1.20, we see that we can represent L in vector form as: $\mathbf{r} + t\mathbf{v}$, for $\mathbf{r} = (-3,1,-4)$ and $\mathbf{v} = (7,3,-2)$. Since the point $Q = (-3,1,-4)$ is on L, then for $\mathbf{w} = \overrightarrow{QP} = (1,1,1) - (-3,1,-4) = (4,0,5)$, we have:

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 7 & 3 & -2 \\ 4 & 0 & 5 \end{vmatrix} = \begin{vmatrix} 3 & -2 \\ 0 & 5 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 7 & -2 \\ 4 & 5 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 7 & 3 \\ 4 & 0 \end{vmatrix}\mathbf{k} = 15\,\mathbf{i} - 43\,\mathbf{j} - 12\,\mathbf{k}\text{ , so}$$

$$d = \frac{\|\mathbf{v} \times \mathbf{w}\|}{\|\mathbf{v}\|} = \frac{\|15\,\mathbf{i} - 43\,\mathbf{j} - 12\,\mathbf{k}\|}{\|(7,3,-2)\|} = \frac{\sqrt{15^2 + (-43)^2 + (-12)^2}}{\sqrt{7^2 + 3^2 + (-2)^2}} = \frac{\sqrt{2218}}{\sqrt{62}} = 5.98$$

It is clear that two lines L_1 and L_2, represented in vector form as $\mathbf{r}_1 + s\mathbf{v}_1$ and $\mathbf{r}_2 + t\mathbf{v}_2$, respectively, are parallel (denoted as $L_1 \parallel L_2$) if $\mathbf{v}_1$ and $\mathbf{v}_2$ are parallel. Also, L_1 and L_2 are perpendicular (denoted as $L_1 \perp L_2$) if $\mathbf{v}_1$ and $\mathbf{v}_2$ are perpendicular.

In 2-dimensional space, two lines are either identical, parallel, or they intersect. In 3-dimensional space, there is an additional possibility: two lines can be **skew**, that is, they do not intersect but they are not parallel. However, even though they are not parallel, skew lines are on parallel planes (see Figure 1.5.5).

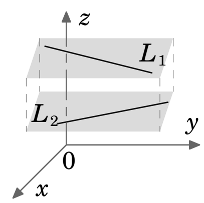


Figure 1.5.5

To determine whether two lines in $\mathbb{R}^3$ intersect, it is often easier to use the parametric representation of the lines. In this case, you should use different parameter variables (usually s and t) for the lines, since the values of the parameters may not be the same at the point of intersection. Setting the two (x,y,z) triples equal will result in a system of 3 equations in 2 unknowns (s and t).

Example 1.22. Find the point of intersection (if any) of the following lines:

$$\frac{x+1}{3} = \frac{y-2}{2} = \frac{z-1}{-1} \quad \text{and} \quad x+3 = \frac{y-8}{-3} = \frac{z+3}{2}$$

Solution: First we write the lines in parametric form, with parameters s and t:

$$x = -1 + 3s, \quad y = 2 + 2s, \quad z = 1 - s \quad \text{and} \quad x = -3 + t, \quad y = 8 - 3t, \quad z = -3 + 2t$$

The lines intersect when $(-1+3s, 2+2s, 1-s) = (-3+t, 8-3t, -3+2t)$ for some s, t:

$$\begin{aligned} -1 + 3s = -3 + t &: \Rightarrow t = 2 + 3s \\ 2 + 2s = 8 - 3t &: \Rightarrow 2 + 2s = 8 - 3(2+3s) = 2 - 9s \Rightarrow 2s = -9s \Rightarrow s = 0 \Rightarrow t = 2 + 3(0) = 2 \\ 1 - s = -3 + 2t &: \; 1 - 0 = -3 + 2(2) \Rightarrow 1 = 1 \;\checkmark \text{ (Note that we had to check this.)} \end{aligned}$$

Letting $s = 0$ in the equations for the first line, or letting $t = 2$ in the equations for the second line, gives the point of intersection $(-1,2,1)$.

We will now consider planes in 3-dimensional Euclidean space.

Plane through a point, perpendicular to a vector

Let P be a plane in $\mathbb{R}^3$, and suppose it contains a point $P_0 = (x_0, y_0, z_0)$. Let $\mathbf{n} = (a, b, c)$ be a nonzero vector which is perpendicular to the plane P. Such a vector is called a **normal vector** (or just a *normal*) to the plane. Now let (x, y, z) be any point in the plane P. Then the vector $\mathbf{r} = (x - x_0, y - y_0, z - z_0)$ lies in the plane P (see Figure 1.5.6). So if $\mathbf{r} \neq \mathbf{0}$, then $\mathbf{r} \perp \mathbf{n}$ and hence $\mathbf{n} \cdot \mathbf{r} = 0$. And if $\mathbf{r} = \mathbf{0}$ then we still have $\mathbf{n} \cdot \mathbf{r} = 0$.

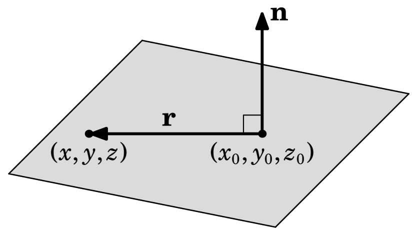


Figure 1.5.6 The plane P

Conversely, if (x, y, z) is any point in $\mathbb{R}^3$ such that $\mathbf{r} = (x - x_0, y - y_0, z - z_0) \neq \mathbf{0}$ and $\mathbf{n} \cdot \mathbf{r} = 0$, then $\mathbf{r} \perp \mathbf{n}$ and so (x, y, z) lies in P. This proves the following theorem:

Theorem 1.18. Let P be a plane in $\mathbb{R}^3$, let (x_0, y_0, z_0) be a point in P, and let $\mathbf{n} = (a, b, c)$ be a nonzero vector which is perpendicular to P. Then P consists of the points (x, y, z) satisfying the vector equation:

$$\mathbf{n} \cdot \mathbf{r} = 0 \tag{1.24}$$

where $\mathbf{r} = (x - x_0, y - y_0, z - z_0)$, or equivalently:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \tag{1.25}$$

The above equation is called the **point-normal form** of the plane P.

Example 1.23. Find the equation of the plane P containing the point $(-3, 1, 3)$ and perpendicular to the vector $\mathbf{n} = (2, 4, 8)$.

Solution: By formula (1.25), the plane P consists of all points (x, y, z) such that:

$$2(x + 3) + 4(y - 1) + 8(z - 3) = 0$$

If we multiply out the terms in formula (1.25) and combine the constant terms, we get an equation of the plane in **normal form**:

$$ax + by + cz + d = 0 \tag{1.26}$$

For example, the normal form of the plane in Example 1.23 is $2x + 4y + 8z - 22 = 0$.

Plane containing three noncollinear points

In 2-dimensional and 3-dimensional space, two points determine a line. Two points do not determine a plane in $\mathbb{R}^3$. In fact, three *collinear* points (i.e. all on the same line) do not determine a plane; an infinite number of planes would contain the line on which those three points lie. However, three *noncollinear* points do determine a plane. For if Q, R and S are noncollinear points in $\mathbb{R}^3$, then $\overrightarrow{QR}$ and $\overrightarrow{QS}$ are nonzero vectors which are not parallel (by noncollinearity), and so their cross product $\overrightarrow{QR} \times \overrightarrow{QS}$ is perpendicular to both $\overrightarrow{QR}$ and $\overrightarrow{QS}$. So $\overrightarrow{QR}$ and $\overrightarrow{QS}$ (and hence Q, R and S) lie in the plane through the point Q with normal vector $\mathbf{n} = \overrightarrow{QR} \times \overrightarrow{QS}$ (see Figure 1.5.7).

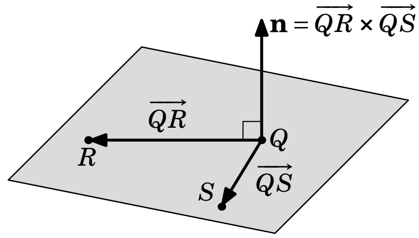


Figure 1.5.7 Noncollinear points Q, R, S

Example 1.24. Find the equation of the plane P containing the points $(2,1,3)$, $(1,-1,2)$ and $(3,2,1)$.

Solution: Let $Q = (2,1,3)$, $R = (1,-1,2)$ and $S = (3,2,1)$. Then for the vectors $\overrightarrow{QR} = (-1,-2,-1)$ and $\overrightarrow{QS} = (1,1,-2)$, the plane P has a normal vector

$$\mathbf{n} = \overrightarrow{QR} \times \overrightarrow{QS} = (-1,-2,-1) \times (1,1,-2) = (5,-3,1)$$

So using formula (1.25) with the point Q (we could also use R or S), the plane P consists of all points (x,y,z) such that:

$$5(x-2) - 3(y-1) + (z-3) = 0$$

or in normal form,

$$5x - 3y + z - 10 = 0$$

We mentioned earlier that skew lines in $\mathbb{R}^3$ lie on separate, parallel planes. So two skew lines do not determine a plane. But two (nonidentical) lines which either intersect or are parallel do determine a plane. In both cases, to find the equation of the plane that contains those two lines, simply pick from the two lines a total of three noncollinear points (i.e. one point from one line and two points from the other), then use the technique above, as in Example 1.24, to write the equation. We will leave examples of this as exercises for the reader.

Distance between a point and a plane

The distance between a point in $\mathbb{R}^3$ and a plane is the length of the line segment from that point to the plane which is perpendicular to the plane. The following theorem gives a formula for that distance.

Theorem 1.19. Let $Q = (x_0, y_0, z_0)$ be a point in $\mathbb{R}^3$, and let P be a plane with normal form $ax + by + cz + d = 0$ that does not contain Q. Then the distance D from Q to P is:

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}} \tag{1.27}$$

Proof: Let $R = (x, y, z)$ be any point in the plane P (so that $ax + by + cz + d = 0$) and let $\mathbf{r} = \overrightarrow{RQ} = (x_0 - x, y_0 - y, z_0 - z)$. Then $\mathbf{r} \neq \mathbf{0}$ since Q does not lie in P. From the normal form equation for P, we know that $\mathbf{n} = (a, b, c)$ is a normal vector for P. Now, any plane divides $\mathbb{R}^3$ into two disjoint parts. Assume that $\mathbf{n}$ points toward the side of P where the point Q is located. Place $\mathbf{n}$ so that its initial point is at R, and let θ be the angle between $\mathbf{r}$ and $\mathbf{n}$. Then $0° < \theta < 90°$, so $\cos\theta > 0$. Thus, the distance D is $\cos\theta \, \|\mathbf{r}\| = |\cos\theta| \, \|\mathbf{r}\|$ (see Figure 1.5.8).

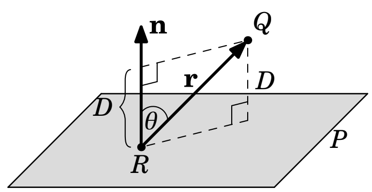


Figure 1.5.8

By Theorem 1.6 in Section 1.3, we know that $\cos\theta = \dfrac{\mathbf{n} \cdot \mathbf{r}}{\|\mathbf{n}\| \, \|\mathbf{r}\|}$, so

$$\begin{aligned} D = |\cos\theta| \, \|\mathbf{r}\| &= \frac{|\mathbf{n} \cdot \mathbf{r}|}{\|\mathbf{n}\| \, \|\mathbf{r}\|} \|\mathbf{r}\| = \frac{|\mathbf{n} \cdot \mathbf{r}|}{\|\mathbf{n}\|} = \frac{|a(x_0 - x) + b(y_0 - y) + c(z_0 - z)|}{\sqrt{a^2 + b^2 + c^2}} \\ &= \frac{|ax_0 + by_0 + cz_0 - (ax + by + cz)|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|ax_0 + by_0 + cz_0 - (-d)|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}} \end{aligned}$$

If $\mathbf{n}$ points away from the side of P where the point Q is located, then $90° < \theta < 180°$ and so $\cos\theta < 0$. The distance D is then $|\cos\theta| \, \|\mathbf{r}\|$, and thus repeating the same argument as above still gives the same result. **QED**

Example 1.25. Find the distance D from $(2, 4, -5)$ to the plane from Example 1.24.

Solution: Recall that the plane is given by $5x - 3y + z - 10 = 0$. So

$$D = \frac{|5(2) - 3(4) + 1(-5) - 10|}{\sqrt{5^2 + (-3)^2 + 1^2}} = \frac{|-17|}{\sqrt{35}} = \frac{17}{\sqrt{35}} \approx 2.87$$

Line of intersection of two planes

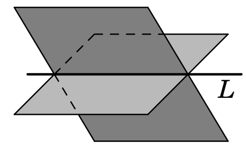


Figure 1.5.9

Note that two planes are parallel if they have normal vectors that are parallel, and the planes are perpendicular if their normal vectors are perpendicular. If two planes do intersect, they do so in a line (see Figure 1.5.9). Suppose that two planes P_1 and P_2 with normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$, respectively, intersect in a line L. Since $\mathbf{n}_1 \times \mathbf{n}_2 \perp \mathbf{n}_1$, then $\mathbf{n}_1 \times \mathbf{n}_2$ is parallel to the plane P_1. Likewise, $\mathbf{n}_1 \times \mathbf{n}_2 \perp \mathbf{n}_2$ means that $\mathbf{n}_1 \times \mathbf{n}_2$ is also parallel to P_2. Thus, $\mathbf{n}_1 \times \mathbf{n}_2$ is parallel to the intersection of P_1 and P_2, i.e. $\mathbf{n}_1 \times \mathbf{n}_2$ is parallel to L. Thus, we can write L in the following vector form:

$$L : \mathbf{r} + t(\mathbf{n}_1 \times \mathbf{n}_2)\ , \ \text{for } -\infty < t < \infty \tag{1.28}$$

where $\mathbf{r}$ is any vector pointing to a point belonging to both planes. To find a point in both planes, find a common solution (x, y, z) to the two normal form equations of the planes. This can often be made easier by setting one of the coordinate variables to zero, which leaves you to solve two equations in just two unknowns.

Example 1.26. Find the line of intersection L of the planes $5x-3y+z-10=0$ and $2x+4y-z+3=0$.

Solution: The plane $5x-3y+z-10=0$ has normal vector $\mathbf{n}_1 = (5,-3,1)$ and the plane $2x+4y-z+3=0$ has normal vector $\mathbf{n}_2 = (2,4,-1)$. Since $\mathbf{n}_1$ and $\mathbf{n}_2$ are not scalar multiples, then the two planes are not parallel and hence will intersect. A point (x, y, z) on both planes will satisfy the following system of two equations in three unknowns:

$$\begin{aligned} 5x-3y+z-10 &= 0 \\ 2x+4y-z+\ \ 3 &= 0 \end{aligned}$$

Set $x = 0$ (why is that a good choice?). Then the above equations are reduced to:

$$\begin{aligned} -3y+z-10 &= 0 \\ 4y-z+\ \ 3 &= 0 \end{aligned}$$

The second equation gives $z = 4y+3$, substituting that into the first equation gives $y = 7$. Then $z = 31$, and so the point $(0,7,31)$ is on L. Since $\mathbf{n}_1 \times \mathbf{n}_2 = (-1,7,26)$, then L is given by:

$$\mathbf{r} + t(\mathbf{n}_1 \times \mathbf{n}_2) = (0,7,31) + t(-1,7,26), \ \ \text{for } -\infty < t < \infty$$

or in parametric form:

$$x = -t, \quad y = 7+7t, \quad z = 31+26t, \ \ \text{for } -\infty < t < \infty$$

Exercises

A

For Exercises 1-4, write the line L through the point P and parallel to the vector $\mathbf{v}$ in the following forms: (a) vector, (b) parametric, and (c) symmetric.

1. $P = (2,3,-2)$, $\mathbf{v} = (5,4,-3)$

2. $P = (3,-1,2)$, $\mathbf{v} = (2,8,1)$

3. $P = (2,1,3)$, $\mathbf{v} = (1,0,1)$

4. $P = (0,0,0)$, $\mathbf{v} = (7,2,-10)$

For Exercises 5-6, write the line L through the points P_1 and P_2 in parametric form.

5. $P_1 = (1,-2,-3)$, $P_2 = (3,5,5)$

6. $P_1 = (4,1,5)$, $P_2 = (-2,1,3)$

For Exercises 7-8, find the distance d from the point P to the line L.

7. $P = (1,-1,-1)$, $L : x = -2-2t,\ y = 4t,\ z = 7+t$

8. $P = (0,0,0)$, $L : x = 3+2t,\ y = 4+3t,\ z = 5+4t$

For Exercises 9-10, find the point of intersection (if any) of the given lines.

9. $x = 7+3s,\ y = -4-3s,\ z = -7-5s$ and $x = 1+6t,\ y = 2+t,\ z = 3-2t$

10. $\dfrac{x-6}{4} = y+3 = z$ and $\dfrac{x-11}{3} = \dfrac{y-14}{-6} = \dfrac{z+9}{2}$

For Exercises 11-12, write the normal form of the plane P containing the point Q and perpendicular to the vector $\mathbf{n}$.

11. $Q = (5,1,-2)$, $\mathbf{n} = (4,-4,3)$

12. $Q = (6,-2,0)$, $\mathbf{n} = (2,6,4)$

For Exercises 13-14, write the normal form of the plane containing the given points.

13. $(1,0,3)$, $(1,2,-1)$, $(6,1,6)$

14. $(-3,1,-3)$, $(4,-4,3)$, $(0,0,1)$

15. Write the normal form of the plane containing the lines from Exercise 9.

16. Write the normal form of the plane containing the lines from Exercise 10.

For Exercises 17-18, find the distance D from the point Q to the plane P.

17. $Q = (4,1,2)$, $P : 3x-y-5z+8 = 0$

18. $Q = (0,2,0)$, $P : -5x+2y-7z+1 = 0$

For Exercises 19-20, find the line of intersection (if any) of the given planes.

19. $x+3y+2z-6 = 0$, $2x-y+z+2 = 0$

20. $3x+y-5z = 0$, $x+2y+z+4 = 0$

B

21. Find the point(s) of intersection (if any) of the line $\dfrac{x-6}{4} = y+3 = z$ with the plane $x+3y+2z-6 = 0$. (*Hint: Put the equations of the line into the equation of the plane.*)

1.6 Surfaces

In the previous section we discussed planes in Euclidean space. A plane is an example of a *surface*, which we will define informally[8] as the solution set of the equation $F(x, y, z) = 0$ in $\mathbb{R}^3$, for some real-valued function F. For example, a plane given by $ax + by + cz + d = 0$ is the solution set of $F(x, y, z) = 0$ for the function $F(x, y, z) = ax + by + cz + d$. Surfaces are 2-dimensional. The plane is the simplest surface, since it is "flat". In this section we will look at some surfaces that are more complex, the most important of which are the sphere and the cylinder.

Definition 1.9. A **sphere** S is the set of all points (x, y, z) in $\mathbb{R}^3$ which are a fixed distance r (called the **radius**) from a fixed point $P_0 = (x_0, y_0, z_0)$ (called the **center** of the sphere):

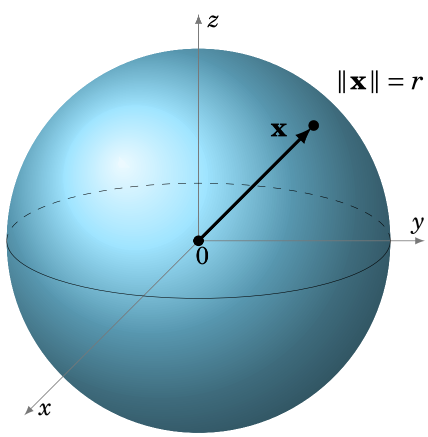
$$S = \{(x, y, z) : (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2\} \tag{1.29}$$

Using vector notation, this can be written in the equivalent form:

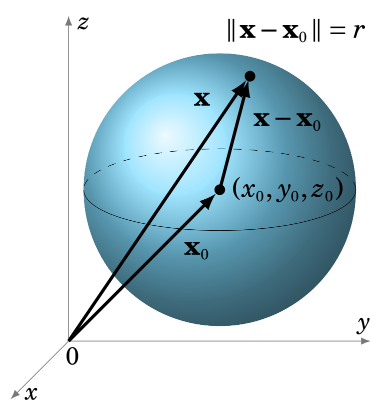
$$S = \{\mathbf{x} : \|\mathbf{x} - \mathbf{x}_0\| = r\} \tag{1.30}$$

where $\mathbf{x} = (x, y, z)$ and $\mathbf{x}_0 = (x_0, y_0, z_0)$ are vectors.

Figure 1.6.1 illustrates the vectorial approach to spheres.



(a) radius r, center $(0, 0, 0)$



(b) radius r, center (x_0, y_0, z_0)

Figure 1.6.1 Spheres in $\mathbb{R}^3$

Note in Figure 1.6.1(a) that the intersection of the sphere with the xy-plane is a circle of radius r (i.e. a *great circle*, given by $x^2 + y^2 = r^2$ as a subset of $\mathbb{R}^2$). Similarly for the intersections with the xz-plane and the yz-plane. In general, a plane intersects a sphere either at a single point or in a circle.

[8] See O'NEILL for a deeper and more rigorous discussion of surfaces.

Example 1.27. Find the intersection of the sphere $x^2 + y^2 + z^2 = 169$ with the plane $z = 12$.

Solution: The sphere is centered at the origin and has radius $13 = \sqrt{169}$, so it does intersect the plane $z = 12$. Putting $z = 12$ into the equation of the sphere gives

$$\begin{aligned} x^2 + y^2 + 12^2 &= 169 \\ x^2 + y^2 &= 169 - 144 = 25 = 5^2 \end{aligned}$$

which is a circle of radius 5 centered at $(0,0,12)$, parallel to the xy-plane (see Figure 1.6.2).

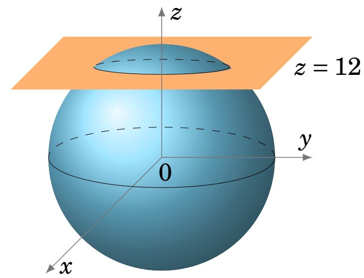


Figure 1.6.2

If the equation in formula (1.29) is multiplied out, we get an equation of the form:

$$x^2 + y^2 + z^2 + ax + by + cz + d = 0 \tag{1.31}$$

for some constants a, b, c and d. Conversely, an equation of this form *may* describe a sphere, which can be determined by completing the square for the x, y and z variables.

Example 1.28. Is $2x^2 + 2y^2 + 2z^2 - 8x + 4y - 16z + 10 = 0$ the equation of a sphere?

Solution: Dividing both sides of the equation by 2 gives

$$\begin{gathered} x^2 + y^2 + z^2 - 4x + 2y - 8z + 5 = 0 \\ (x^2 - 4x + 4) + (y^2 + 2y + 1) + (z^2 - 8z + 16) + 5 - 4 - 1 - 16 = 0 \\ (x-2)^2 + (y+1)^2 + (z-4)^2 = 16 \end{gathered}$$

which is a sphere of radius 4 centered at $(2,-1,4)$.

Example 1.29. Find the points(s) of intersection (if any) of the sphere from Example 1.28 and the line $x = 3 + t$, $y = 1 + 2t$, $z = 3 - t$.

Solution: Put the equations of the line into the equation of the sphere, which was $(x-2)^2 + (y+1)^2 + (z-4)^2 = 16$, and solve for t:

$$\begin{aligned} (3 + t - 2)^2 + (1 + 2t + 1)^2 + (3 - t - 4)^2 &= 16 \\ (t+1)^2 + (2t+2)^2 + (-t-1)^2 &= 16 \\ 6t^2 + 12t - 10 &= 0 \end{aligned}$$

The quadratic formula gives the solutions $t = -1 \pm \dfrac{4}{\sqrt{6}}$. Putting those two values into the equations of the line gives the following two points of intersection:

$$\left(2 + \frac{4}{\sqrt{6}}, -1 + \frac{8}{\sqrt{6}}, 4 - \frac{4}{\sqrt{6}}\right) \quad \text{and} \quad \left(2 - \frac{4}{\sqrt{6}}, -1 - \frac{8}{\sqrt{6}}, 4 + \frac{4}{\sqrt{6}}\right)$$

If two spheres intersect, they do so either at a single point or in a circle.

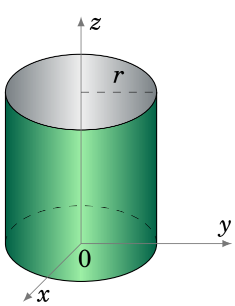
Example 1.30. Find the intersection (if any) of the spheres $x^2+y^2+z^2=25$ and $x^2+y^2+(z-2)^2=16$.

Solution: For any point (x,y,z) on both spheres, we see that

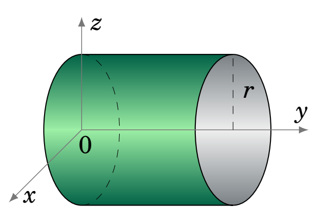
$$\begin{aligned} x^2+y^2+z^2=25 \quad &\Rightarrow \quad x^2+y^2=25-z^2, \text{ and} \\ x^2+y^2+(z-2)^2=16 \quad &\Rightarrow \quad x^2+y^2=16-(z-2)^2, \text{ so} \\ 16-(z-2)^2=25-z^2 \quad &\Rightarrow \quad 4z-4=9 \quad \Rightarrow \quad z=13/4 \\ &\Rightarrow \quad x^2+y^2=25-(13/4)^2=231/16 \end{aligned}$$

$\therefore$ The intersection is the circle $x^2+y^2=\frac{231}{16}$ of radius $\frac{\sqrt{231}}{4}\approx 3.8$ centered at $(0,0,\frac{13}{4})$.

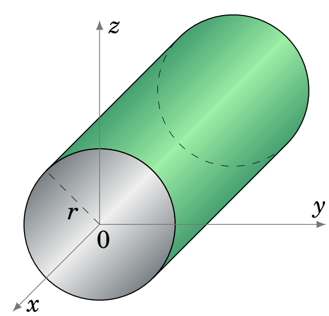
The cylinders that we will consider are *right circular cylinders*. These are cylinders obtained by moving a line L along a circle C in $\mathbb{R}^3$ in a way so that L is always perpendicular to the plane containing C. We will only consider the cases where the plane containing C is parallel to one of the three coordinate planes (see Figure 1.6.3).



(a) $x^2+y^2=r^2$, any z



(b) $x^2+z^2=r^2$, any y



(c) $y^2+z^2=r^2$, any x

Figure 1.6.3 Cylinders in $\mathbb{R}^3$

For example, the equation of a cylinder whose base circle C lies in the xy-plane and is centered at $(a,b,0)$ and has radius r is

$$(x-a)^2+(y-b)^2=r^2, \tag{1.32}$$

where the value of the z coordinate is unrestricted. Similar equations can be written when the base circle lies in one of the other coordinate planes. A plane intersects a right circular cylinder in a circle, ellipse, or one or two lines, depending on whether that plane is parallel, oblique[9], or perpendicular, respectively, to the plane containing C. The intersection of a surface with a plane is called the **trace** of the surface.

[9] i.e. at an angle strictly between $0°$ and $90°$.

The equations of spheres and cylinders are examples of *second-degree equations* in $\mathbb{R}^3$, i.e. equations of the form

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0 \tag{1.33}$$

for some constants $A, B, \dots, J$. If the above equation is not that of a sphere, cylinder, plane, line or point, then the resulting surface is called a **quadric surface**.

One type of quadric surface is the **ellipsoid**, given by an equation of the form:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{1.34}$$

In the case where $a = b = c$, this is just a sphere. In general, an ellipsoid is egg-shaped (think of an ellipse rotated around its major axis). Its traces in the coordinate planes are ellipses.

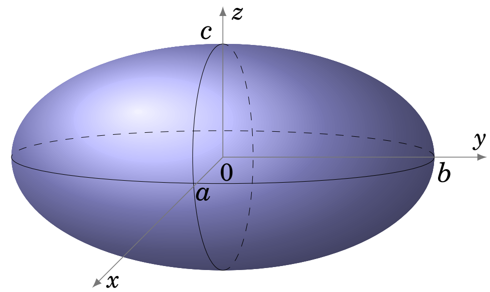


Figure 1.6.4 Ellipsoid

Two other types of quadric surfaces are the **hyperboloid of one sheet**, given by an equation of the form:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \tag{1.35}$$

and the **hyperboloid of two sheets**, whose equation has the form:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \tag{1.36}$$

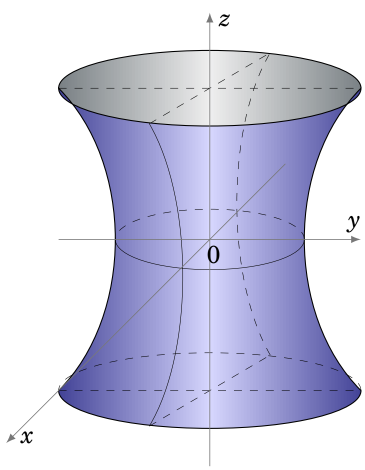


Figure 1.6.5 Hyperboloid of one sheet

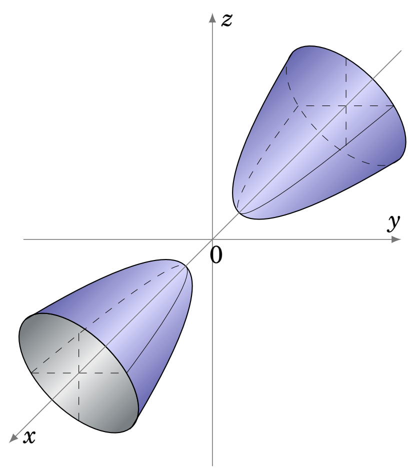


Figure 1.6.6 Hyperboloid of two sheets

For the hyperboloid of one sheet, the trace in any plane parallel to the xy-plane is an ellipse. The traces in the planes parallel to the xz- or yz-planes are hyperbolas (see Figure 1.6.5), except for the special cases $x = \pm a$ and $y = \pm b$; in those planes the traces are pairs of intersecting lines (see Exercise 8).

For the hyperboloid of two sheets, the trace in any plane parallel to the xy- or xz-plane is a hyperbola (see Figure 1.6.6). There is no trace in the yz-plane. In any plane parallel to the yz-plane for which $|x| > |a|$, the trace is an ellipse.

The **elliptic paraboloid** is another type of quadric surface, whose equation has the form:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c} \tag{1.37}$$

The traces in planes parallel to the xy-plane are ellipses, though in the xy-plane itself the trace is a single point. The traces in planes parallel to the xz- or yz-planes are parabolas. Figure 1.6.7 shows the case where $c > 0$. When $c < 0$ the surface is turned downward. In the case where $a = b$, the surface is called a *paraboloid of revolution*, which is often used as a reflecting surface, e.g. in vehicle headlights.[10]

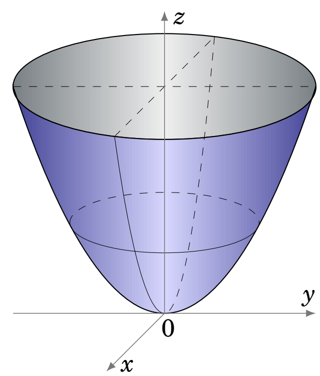


Figure 1.6.7 Paraboloid

A more complicated quadric surface is the **hyperbolic paraboloid**, given by:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c} \tag{1.38}$$

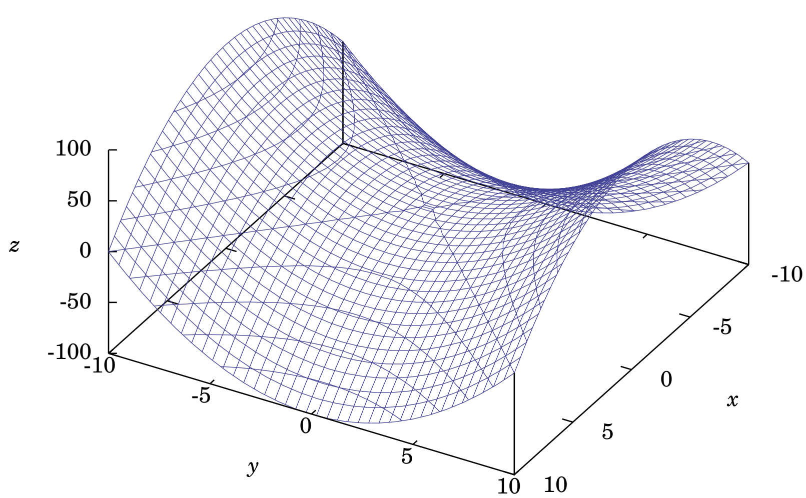


Figure 1.6.8 Hyperbolic paraboloid

[10] For a discussion of this see pp. 157-158 in HECHT.

The hyperbolic paraboloid can be tricky to draw; using graphing software on a computer can make it easier. For example, Figure 1.6.8 was created using the free Gnuplot package (see Appendix C). It shows the graph of the hyperbolic paraboloid $z = y^2 - x^2$, which is the special case where $a = b = 1$ and $c = -1$ in equation (1.38). The mesh lines on the surface are the traces in planes parallel to the coordinate planes. So we see that the traces in planes parallel to the xz-plane are parabolas pointing upward, while the traces in planes parallel to the yz-plane are parabolas pointing downward. Also, notice that the traces in planes parallel to the xy-plane are hyperbolas, though in the xy-plane itself the trace is a pair of intersecting lines through the origin. This is true in general when $c < 0$ in equation (1.38). When $c > 0$, the surface would be similar to that in Figure 1.6.8, only rotated 90° around the z-axis and the nature of the traces in planes parallel to the xz- or yz-planes would be reversed.

The last type of quadric surface that we will consider is the **elliptic cone**, which has an equation of the form:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0 \tag{1.39}$$

The traces in planes parallel to the xy-plane are ellipses, except in the xy-plane itself where the trace is a single point. The traces in planes parallel to the xz- or yz-planes are hyperbolas, except in the xz- and yz-planes themselves where the traces are pairs of intersecting lines.

Notice that every point on the elliptic cone is on a line which lies entirely on the surface; in Figure 1.6.9 these lines all go through the origin. This makes the elliptic cone an example of a *ruled surface*. The cylinder is also a ruled surface.

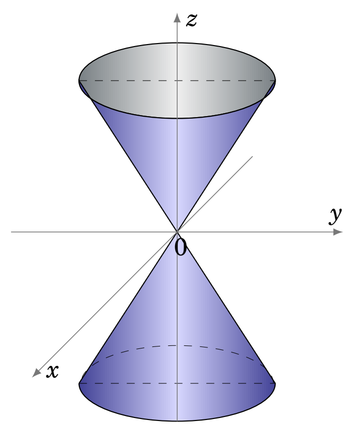


Figure 1.6.9 Elliptic cone

What may not be as obvious is that both the hyperboloid of one sheet and the hyperbolic paraboloid are ruled surfaces. In fact, on both surfaces there are *two* lines through each point on the surface (see Exercises 11-12). Such surfaces are called *doubly ruled surfaces*, and the pairs of lines are called a *regulus*.

It is clear that for each of the six types of quadric surfaces that we discussed, the surface can be translated away from the origin (e.g. by replacing x^2 by $(x-x_0)^2$ in its equation). It can be proved[11] that *every* quadric surface can be translated and/or rotated so that its equation matches one of the six types that we described. For example, $z = 2xy$ is a case of equation (1.33) with “mixed” variables, e.g. with $D \neq 0$ so that we get an xy term. This equation does not match any of the types we considered. However, by rotating the x- and y-axes by 45° in the xy-plane by means of the coordinate transformation $x = (x'-y')/\sqrt{2}$, $y = (x'+y')/\sqrt{2}$, $z = z'$, then $z = 2xy$ becomes the hyperbolic paraboloid $z' = (x')^2 - (y')^2$ in the (x', y', z') coordinate system. That is, $z = 2xy$ is a hyperbolic paraboloid as in equation (1.38), but rotated 45° in the xy-plane.

[11]See Ch. 7 in POGORELOV.

Exercises

A

For Exercises 1-4, determine if the given equation describes a sphere. If so, find its radius and center.

1. $x^2+y^2+z^2-4x-6y-10z+37=0$

2. $x^2+y^2+z^2+2x-2y-8z+19=0$

3. $2x^2+2y^2+2z^2+4x+4y+4z-44=0$

4. $x^2+y^2-z^2+12x+2y-4z+32=0$

5. Find the point(s) of intersection of the sphere $(x-3)^2+(y+1)^2+(z-3)^2=9$ and the line $x=-1+2t,\ y=-2-3t,\ z=3+t$.

B

6. Find the intersection of the spheres $x^2+y^2+z^2=9$ and $(x-4)^2+(y+2)^2+(z-4)^2=9$.

7. Find the intersection of the sphere $x^2+y^2+z^2=9$ and the cylinder $x^2+y^2=4$.

8. Find the trace of the hyperboloid of one sheet $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ in the plane $x=a$, and the trace in the plane $y=b$.

9. Find the trace of the hyperbolic paraboloid $\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{z}{c}$ in the xy-plane.

C

10. It can be shown that any four *noncoplanar* points (i.e. points that do not lie in the same plane) determine a sphere.[12] Find the equation of the sphere that passes through the points $(0,0,0)$, $(0,0,2)$, $(1,-4,3)$ and $(0,-1,3)$. *(Hint: Equation (1.31))*

11. Show that the hyperboloid of one sheet is a doubly ruled surface, i.e. each point on the surface is on two lines lying entirely on the surface. *(Hint: Write equation (1.35) as* $\frac{x^2}{a^2}-\frac{z^2}{c^2}=1-\frac{y^2}{b^2}$ *, factor each side. Recall that two planes intersect in a line.)*

12. Show that the hyperbolic paraboloid is a doubly ruled surface. *(Hint: Exercise 11)*

13. Let S be the sphere with radius 1 centered at $(0,0,1)$, and let S^* be S without the "north pole" point $(0,0,2)$. Let (a,b,c) be an arbitrary point on S^*. Then the line passing through $(0,0,2)$ and (a,b,c) intersects the xy-plane at some point $(x,y,0)$, as in Figure 1.6.10. Find this point $(x,y,0)$ in terms of a, b and c.
(Note: Every point in the xy-plane can be matched with a point on S^*, and vice versa, in this manner. This method is called *stereographic projection*, which essentially identifies all of $\mathbb{R}^2$ with a "punctured" sphere.)

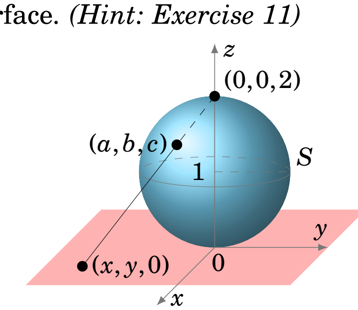


Figure 1.6.10

[12]See WELCHONS and KRICKENBERGER, p. 160, for a proof.

1.7 Curvilinear Coordinates

The Cartesian coordinates of a point (x, y, z) are determined by following straight paths starting from the origin: first along the x-axis, then parallel to the y-axis, then parallel to the z-axis, as in Figure 1.7.1. In *curvilinear coordinate systems*, these paths can be curved. The two types of curvilinear coordinates which we will consider are cylindrical and spherical coordinates. Instead of referencing a point in terms of sides of a rectangular parallelepiped, as with Cartesian coordinates, we will think of the point as lying on a cylinder or sphere. Cylindrical coordinates are often used when there is symmetry around the z-axis; spherical coordinates are useful when there is symmetry about the origin.

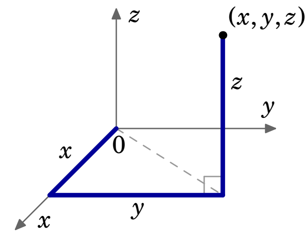


Figure 1.7.1

Let $P = (x, y, z)$ be a point in Cartesian coordinates in $\mathbb{R}^3$, and let $P_0 = (x, y, 0)$ be the projection of P upon the xy-plane. Treating (x, y) as a point in $\mathbb{R}^2$, let (r, θ) be its polar coordinates (see Figure 1.7.2). Let ρ be the length of the line segment from the origin to P, and let ϕ be the angle between that line segment and the positive z-axis (see Figure 1.7.3). ϕ is called the *zenith angle*. Then the **cylindrical coordinates** (r, θ, z) and the **spherical coordinates** (ρ, θ, ϕ) of $P(x, y, z)$ are defined as follows:[13]

Cylindrical coordinates (r, θ, z):

$$x = r\cos\theta \qquad r = \sqrt{x^2 + y^2}$$
$$y = r\sin\theta \qquad \theta = \tan^{-1}\left(\tfrac{y}{x}\right)$$
$$z = z \qquad z = z$$

where $0 \le \theta \le \pi$ if $y \ge 0$ and $\pi < \theta < 2\pi$ if $y < 0$

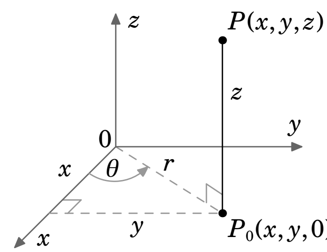


Figure 1.7.2
Cylindrical coordinates

Spherical coordinates (ρ, θ, ϕ):

$$x = \rho\sin\phi\cos\theta \qquad \rho = \sqrt{x^2 + y^2 + z^2}$$
$$y = \rho\sin\phi\sin\theta \qquad \theta = \tan^{-1}\left(\tfrac{y}{x}\right)$$
$$z = \rho\cos\phi \qquad \phi = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

where $0 \le \theta \le \pi$ if $y \ge 0$ and $\pi < \theta < 2\pi$ if $y < 0$

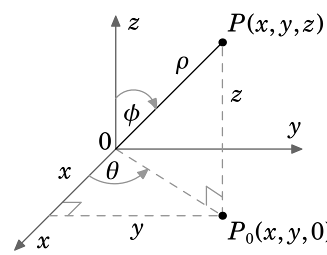


Figure 1.7.3
Spherical coordinates

Both θ and ϕ are measured in radians. Note that $r \ge 0$, $0 \le \theta < 2\pi$, $\rho \ge 0$ and $0 \le \phi \le \pi$. Also, θ is undefined when $(x, y) = (0, 0)$, and ϕ is undefined when $(x, y, z) = (0, 0, 0)$.

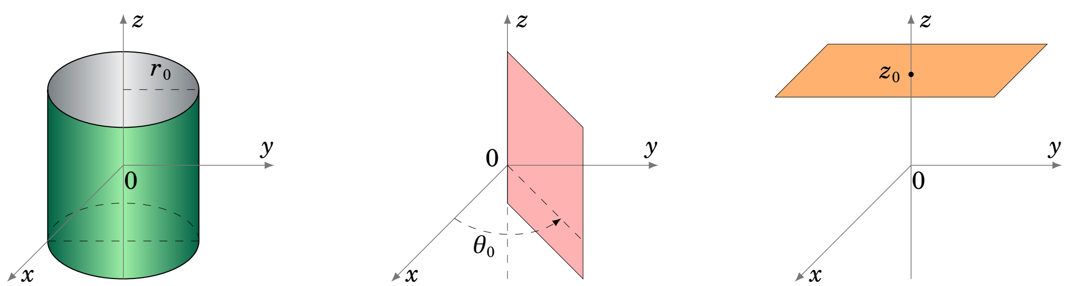
[13]This "standard" definition of spherical coordinates used by mathematicians results in a left-handed system. For this reason, physicists usually switch the definitions of θ and ϕ to make (ρ, θ, ϕ) a right-handed system.

Example 1.31. Convert the point $(-2,-2,1)$ from Cartesian coordinates to (a) cylindrical and (b) spherical coordinates.

Solution: (a) $r = \sqrt{(-2)^2+(-2)^2} = 2\sqrt{2}$, $\theta = \tan^{-1}\left(\frac{-2}{-2}\right) = \tan^{-1}(1) = \frac{5\pi}{4}$, since $y = -2 < 0$.
$\therefore\ (r,\theta,z) = \left(2\sqrt{2}, \frac{5\pi}{4}, 1\right)$

(b) $\rho = \sqrt{(-2)^2+(-2)^2+1^2} = \sqrt{9} = 3$, $\phi = \cos^{-1}\left(\frac{1}{3}\right) \approx 1.23$ radians.
$\therefore\ (\rho,\theta,\phi) = \left(3, \frac{5\pi}{4}, 1.23\right)$

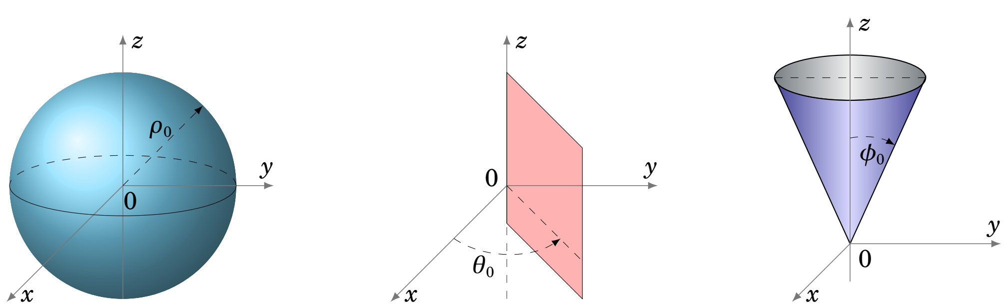
For cylindrical coordinates (r,θ,z), and constants r_0, θ_0 and z_0, we see from Figure 1.7.4 that the surface $r = r_0$ is a cylinder of radius r_0 centered along the z-axis, the surface $\theta = \theta_0$ is a half-plane emanating from the z-axis, and the surface $z = z_0$ is a plane parallel to the xy-plane.



(a) $r = r_0$ **(b)** $\theta = \theta_0$ **(c)** $z = z_0$

Figure 1.7.4 Cylindrical coordinate surfaces

For spherical coordinates (ρ,θ,ϕ), and constants ρ_0, θ_0 and ϕ_0, we see from Figure 1.7.5 that the surface $\rho = \rho_0$ is a sphere of radius ρ_0 centered at the origin, the surface $\theta = \theta_0$ is a half-plane emanating from the z-axis, and the surface $\phi = \phi_0$ is a circular cone whose vertex is at the origin.



(a) $\rho = \rho_0$ **(b)** $\theta = \theta_0$ **(c)** $\phi = \phi_0$

Figure 1.7.5 Spherical coordinate surfaces

Figures 1.7.4(a) and 1.7.5(a) show how these coordinate systems got their names.

Sometimes the equation of a surface in Cartesian coordinates can be transformed into a simpler equation in some other coordinate system, as in the following example.

Example 1.32. Write the equation of the cylinder $x^2 + y^2 = 4$ in cylindrical coordinates.

Solution: Since $r = \sqrt{x^2 + y^2}$, then the equation in cylindrical coordinates is $r = 2$.

Using spherical coordinates to write the equation of a sphere does not necessarily make the equation simpler, if the sphere is not centered at the origin.

Example 1.33. Write the equation $(x-2)^2 + (y-1)^2 + z^2 = 9$ in spherical coordinates.

Solution: Multiplying the equation out gives

$$\begin{aligned} x^2 + y^2 + z^2 - 4x - 2y + 5 = 9 &\text{ , so we get} \\ \rho^2 - 4\rho\sin\phi\cos\theta - 2\rho\sin\phi\sin\theta - 4 = 0 &\text{ , or} \\ \rho^2 - 2\sin\phi(2\cos\theta - \sin\theta)\rho - 4 = 0 & \end{aligned}$$

after combining terms. Note that this actually makes it more difficult to figure out what the surface is, as opposed to the Cartesian equation where you could immediately identify the surface as a sphere of radius 3 centered at $(2,1,0)$.

Example 1.34. Describe the surface given by $\theta = z$ in cylindrical coordinates.

Solution: This surface is called a *helicoid*. As the (vertical) z coordinate increases, so does the angle θ, while the radius r is unrestricted. So this sweeps out a (ruled!) surface shaped like a spiral staircase, where the spiral has an infinite radius. Figure 1.7.6 shows a section of this surface restricted to $0 \le z \le 4\pi$ and $0 \le r \le 2$.

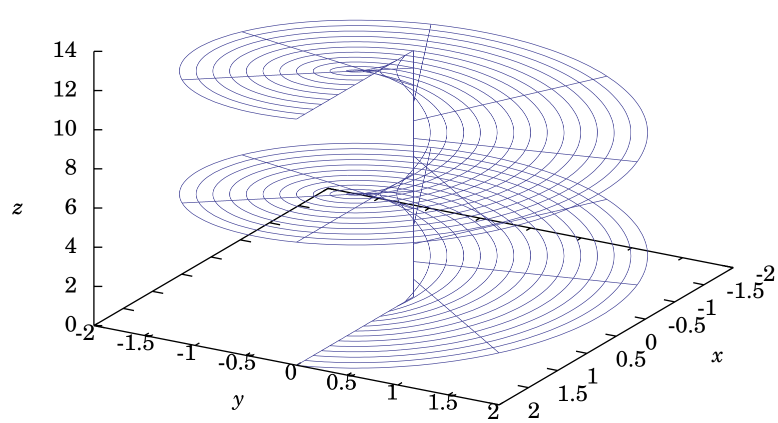


Figure 1.7.6 Helicoid $\theta = z$

Exercises

A

For Exercises 1-4, find the (a) cylindrical and (b) spherical coordinates of the point whose Cartesian coordinates are given.

1. $(2, 2\sqrt{3}, -1)$ **2.** $(-5, 5, 6)$ **3.** $(\sqrt{21}, -\sqrt{7}, 0)$ **4.** $(0, \sqrt{2}, 2)$

For Exercises 5-7, write the given equation in (a) cylindrical and (b) spherical coordinates.

5. $x^2 + y^2 + z^2 = 25$ **6.** $x^2 + y^2 = 2y$ **7.** $x^2 + y^2 + 9z^2 = 36$

B

8. Describe the intersection of the surfaces whose equations in spherical coordinates are $\theta = \frac{\pi}{2}$ and $\phi = \frac{\pi}{4}$.

9. Show that for $a \neq 0$, the equation $\rho = 2a \sin\phi \cos\theta$ in spherical coordinates describes a sphere centered at $(a, 0, 0)$ with radius $|a|$.

C

10. Let $P = (a, \theta, \phi)$ be a point in spherical coordinates, with $a > 0$ and $0 < \phi < \pi$. Then P lies on the sphere $\rho = a$. Since $0 < \phi < \pi$, the line segment from the origin to P can be extended to intersect the cylinder given by $r = a$ (in cylindrical coordinates). Find the cylindrical coordinates of that point of intersection.

11. Let P_1 and P_2 be points whose spherical coordinates are $(\rho_1, \theta_1, \phi_1)$ and $(\rho_2, \theta_2, \phi_2)$, respectively. Let $\mathbf{v}_1$ be the vector from the origin to P_1, and let $\mathbf{v}_2$ be the vector from the origin to P_2. For the angle γ between $\mathbf{v}_1$ and $\mathbf{v}_2$, show that

$$\cos\gamma = \cos\phi_1 \cos\phi_2 + \sin\phi_1 \sin\phi_2 \cos(\theta_2 - \theta_1).$$

This formula is used in electrodynamics to prove the addition theorem for spherical harmonics, which provides a general expression for the electrostatic potential at a point due to a unit charge. See pp. 100-102 in JACKSON.

12. Show that the distance d between the points P_1 and P_2 with cylindrical coordinates (r_1, θ_1, z_1) and (r_2, θ_2, z_2), respectively, is

$$d = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_2 - \theta_1) + (z_2 - z_1)^2}.$$

13. Show that the distance d between the points P_1 and P_2 with spherical coordinates $(\rho_1, \theta_1, \phi_1)$ and $(\rho_2, \theta_2, \phi_2)$, respectively, is

$$d = \sqrt{\rho_1^2 + \rho_2^2 - 2\rho_1 \rho_2 [\sin\phi_1 \sin\phi_2 \cos(\theta_2 - \theta_1) + \cos\phi_1 \cos\phi_2]}.$$